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Critical Behavior and Associated Conformal Algebra of the Z_3 Potts Model

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The conformal algebra for operators of the Z_3 model at the phase transition point is built. Critical exponents are found in this approach as solutions of simple algebraic equations, which are consistency conditions of the algebra. Multipoint correlation functions obey linear differential equations. Some solutions are given for the four-point correlation functions of the Z_3 model at the phase transition point.

KEY WORDS: Conformal algebra; operators; dimensions; critical behavior of two-dimensional Z_3 model; differential equations for multipoint correlators.

The partition function of the two-dimensional (2D) Z_3 model of lattice statistics, also called the three-component Potts model,[1] can be defined as follows:

$$Z(\beta) = \sum_{\{\sigma\}} \exp\left(+\beta \sum_{x,\alpha} \frac{\sigma_x \bar{\sigma}_{x+\alpha} + \bar{\sigma}_x \sigma_{x+\alpha}}{2} \right)$$

$$= \sum_{\{\sigma\}} \exp\left[\beta \sum_{x,\alpha} \cos(\varphi_x - \varphi_{x+\alpha}) \right] \tag{1}$$

Its discrete spin variables $[\sigma = \exp(i\varphi),\ \bar{\sigma} = \exp(-i\varphi);\ \varphi = 0, \pm 2\pi/3]$ interacting with nearest neighbors only, are placed at sites of a 2D lattice;

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$x = (n, m)$ are lattice sites; $\alpha \equiv \boldsymbol{\alpha} = \mathbf{1}, \mathbf{2}$ are two basic vectors of the lattice —for simplicity we can consider a square lattice. This model is known to be self-dual, the same as the 2D Ising model (IM). At its self-dual point $\beta_c = \frac{2}{3}\ln(\sqrt{3} + 1)$ it undergoes a second-order phase transition.[2]

At present it is almost universally accepted [3] that critical properties of the Z_3 model are the same as those of the hard hexagon model (HHM), which has been solved recently by Baxter,[4] and thereby the critical exponents of the model are thought to be known exactly.

It is also known that the Z_3 model itself becomes soluble (integrable) just at the phase transition point.[2] It raises the possibility that a continuum theory should exist, simpler than the lattice one, to which the lattice model reduces at the critical point, and which would be solvable by continuum theory methods, making it possible to study exactly the critical behavior of the model (its critical exponents and critical multipoint correlators).

It has been suggested some time ago[5] that critical fluctuations in statistical system are not only scale invariant but also conformal invariant, which is a sort of localization of scaling symmetry. Yet for general dimensions of space, conformal transformations make only a finite parameter group. It has been shown in Ref. 5 that conformal symmetry fixes the form of the three-point correlators, in addition to two-point ones which are fixed by scaling symmetry.

The situation is drastically different for 2D systems. In two dimensions the "small" conformal group can be extended to the infinite-parameter general conformal group, given by all analytic transformations.[6] Being infinite dimensional this symmetry could provide "enough means" for solving 2D conformal invariant systems exactly. Different theories would arise as different representations of the corresponding conformal algebra. It has been shown in Ref. 6 how 2D IM can be solved in this way. In this paper we present the conformal algebra of the Z_3 model and study some of its basic properties.

From now on by the Z_3 model we mean its scaling limit theory at the phase transition point. In fact our problem will be to build this theory and solve it.

The first question is what are the basic operators of the model. It is natural to have operators $\sigma(x)$ and $\bar{\sigma}(x)$, which are the scaling limit of lattice spin variables in (1), and, also, the energy operator $\epsilon(x)$, which results from the scaling limit of the interaction term $\sigma_x \bar{\sigma}_{x'} + \bar{\sigma}_x \sigma_{x'}$ in the exponent of (1). All operators will be defined so that their statistical average is zero: $\langle\sigma\rangle = \langle\bar{\sigma}\rangle = \langle\epsilon\rangle = 0$. From their lattice origin and usual scaling considerations it is natural to expect the following operator algebra relations for $\sigma, \bar{\sigma}, \epsilon$:

$$
\begin{aligned}
\sigma(x)\bar{\sigma}(x') + \bar{\sigma}(x)\sigma(x') &\sim \frac{1}{|x-x'|} 2\Delta_\sigma \cdot I + \frac{1}{|x-x'|} 2\Delta_\sigma - \Delta_\epsilon \cdot \epsilon(x') + \cdots \\
\epsilon(x)\sigma(x') &\sim \frac{1}{|x-x'|} \Delta_\epsilon \cdot \sigma(x') + \cdots \\
\epsilon(x)\epsilon(x') &\sim \frac{1}{|x-x'|} 2\Delta_\epsilon \cdot I + \cdots
\end{aligned}
\tag{2}
$$

Here I is a unit (identity) operator of the algebra $[I \cdot \sigma(x) = \sigma(x)$, so on, $\langle I \rangle = 1]$, and Δ_σ and Δ_ϵ are the critical dimensions of operators σ, ϵ.

In the following we shall often write such operator relations in a compact form skipping the standard scaling factors:

$$
\begin{gathered}
\sigma\bar{\sigma} + \bar{\sigma}\sigma \sim I + \epsilon + \cdots \\
\epsilon\sigma \sim \sigma + \cdots, \qquad \epsilon\bar{\sigma} \sim \bar{\sigma} + \cdots, \qquad \epsilon\epsilon \sim I + \cdots \\
\sigma\sigma \sim \bar{\sigma} + \cdots, \qquad \bar{\sigma}\bar{\sigma} \sim \sigma + \cdots
\end{gathered}
\tag{2'}
$$

Thus, (2′) are the operator algebra relations which we should expect from the correct conformal theory.

Conformal invariant theory should always possess one more operator —the energy-momentum tensor $T_{ab}(x)$, which is related to conformal transformations. In 2D conformal theory it is more convenient to use complex coordinates ($z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$). Infinitesimal conformal transformations are defined as

$$
\begin{gathered}
z \to z + \alpha(z), \qquad \bar{z} \to \bar{z} + \bar{\alpha}(z) \\
\psi(z,\bar{z}) \to \psi(z,\bar{z}) + \left[\alpha(z)\partial_z + \alpha'(z)\Delta + \bar{\alpha}(z)\bar{\partial}_z + \bar{\alpha}'(z)\bar{\Delta} \right] \psi(z,\bar{z})
\end{gathered}
\tag{3}
$$

where ψ is a conformal operator; $\Delta, \bar{\Delta}$ are its conformal dimensions; $\alpha(z)$ is an arbitrary analytic function (small, in a finite region of the z plane):

$$
\alpha(z) = \sum_{n=-1}^{\infty} a_n z^{n+1} \tag{4}
$$

The infinitesimal parameters are $\{a_n\}$. Choosing $\alpha(z) = a = \text{const}$, and $\alpha(z) = a \cdot (z - z_0)$ it is easy to show that for an arbitrary correlator $\langle \psi_1 \psi_2 \cdots \rangle$

$$
\partial_{\bar{z}} \langle T_{zz}(z,\bar{z}) \psi_1(z_1\bar{z}_1) \psi_2(z_2\bar{z}_2) \cdots \rangle \tag{5}
$$

$$
\langle T_{z\bar{z}}(z,\bar{z}) \psi_1(z_1\bar{z}_1) \psi_2(z_2\bar{z}_2) \cdots \rangle \tag{6}
$$

if $z \neq z_i$. Equation (5) shows that correlator $\langle T_{zz} \psi_1 \psi_2 \cdots \rangle$ is an analytic function of z only, for $z \neq z_i$. For general $\alpha(z)$, correlators of a conformal invariant theory satisfy the following relations:

$$\oint_C d\xi\, \alpha(\xi)\langle T_{zz}(\xi)\psi_1 \ldots\rangle + \oint_C d\bar{\xi}\, \bar{\alpha}(\xi)\langle T_{\bar{z}\bar{z}}(\bar{\xi})\psi_1 \ldots\rangle$$
$$= \sum_i \left[\alpha(z_i)\partial_i + \Delta_i\alpha'(z_i) + \bar{\alpha}(z_i)\bar{\partial}_i + \bar{\Delta}_i\bar{\alpha}'(z_i)\right]\langle\psi_1 \ldots\rangle \qquad (7)$$

Relation (7) shows that the conformal transformation splits into independent z and $\bar{z}$ parts. This fact allows the formal reduction of 2D conformal invariant theories to 1D ones, dependent on z only. Conformal invariance ensures that dependence on $\bar{z}$ in final expresssions can easily be restored using symmetry considerations and some other requirements, as, e.g., the reality condition on correlators of real operators. Some examples of recovering physical correlators out of conformal ones will be given at the end of the paper, but now we suppress all $\bar{z}$ dependence altogether. Note that the conformal dimension Δ of real operators should be one half of its physical critical dimension Δ_{ph}:

$$\langle\psi(z,\bar{z})\psi(z',\bar{z}')\rangle \sim \frac{1}{|z-z'|}\,2\Delta_{\text{ph}} = \frac{1}{(z-z')^{\Delta_{\text{ph}}}(\bar{z}-\bar{z}')}\,\Delta_{\text{ph}}$$

which becomes

$$\langle\psi(z)\psi(z')\rangle \sim \frac{1}{(z-z')}\,\Delta_{\text{ph}} \equiv \frac{1}{(z-z')}\,2\Delta$$

if $\bar{z}$ dependence is suppressed. Finally (7) becomes

$$\oint_C dz\, \alpha(z)\langle T(z)\psi_1(z_1) \ldots\rangle = \sum_i \left(\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_i\right)\langle\psi_1 \ldots\rangle \qquad (8)$$

Because $\alpha(z)$ is an arbitrary function along C Eq. (8) can be rewritten as[6]

$$\langle T(z)\psi_1 \ldots\rangle = \sum_i \left(\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_i\right)\langle\psi_1 \ldots\rangle \qquad (9)$$

which is the conformal Ward identity (WI).

To derive a closed equation for the correlator $\langle\psi_1 \ldots\rangle$ in the WI (9), one needs the operator expansion for $T(z)\psi(z_1)$, as $z \to z_1$:

$$T(z)\psi(z_1) \approx \frac{\Delta_1}{(z-z_1)^2}\psi(z_1) + \frac{1}{z-z_1}\partial_1\psi(z_1)$$
$$+\psi^{(-2)}(z_1) + (z-z_1)\psi^{(-3)}(z_1) + \cdots \qquad (10)$$

Expansion in integer powers of $(z - z_1)$ is ensured by analyticity of $\langle T(z)\psi_1 \ldots\rangle$. The first two terms in (10) are fixed by (9). Operators standing at higher powers of $(z - z_1)$ can contain corresponding derivatives of ψ and, in principle, new operators: $\psi^{(-2)}(z_1) = C_2\partial_1^2\psi(z_1) + \chi_2(z_1)$. Suppose for a moment that $\chi_2 = 0$ and see what theory will result. Multiplying (10) by $\psi(z_2)$ and taking the average we obtain

$$\langle T(z)\psi(z_1)\psi(z_2)\rangle_{z\to z_1} \approx \left[\frac{\Delta_1}{(z-z_1)^2} + \frac{1}{z-z_1}\partial_1 + C_2\partial_1^2 \right] \frac{1}{(z_1-z_2)^{2\Delta}} + O(z-z_1)$$

Comparing this expansion with the WI (9) written for the correlator $\langle\psi(z_1)\psi(z_2)\rangle$ we find $C_2 = 3/2(2\Delta_1 + 1)$. Now, for arbitrary correlator $\langle\psi_1 \ldots\rangle$ we have

$$\langle T(z)\psi_1(z_1)\ldots\rangle_{z\to z_1} \approx \left[\frac{\Delta_1}{(z-z_1)^2} + \frac{1}{z-z_1}\partial_1 + \frac{3}{2(2\Delta_1+1)}\partial_1^2 \right] \langle\psi_1 \ldots\rangle + O(z-z_1) \tag{11}$$

Again, comparing (11) with the corresponding expansion derived from WI (9) we find a closed linear differential equation for the conformal correlator $\langle\psi_1 \ldots\rangle$:[6]

$$\frac{3}{2(2\Delta_1+1)}\partial_1^2\langle\psi_1 \ldots\rangle = \sum_{i\neq 1}\left[\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_i \right]\langle\psi_1 \ldots\rangle \tag{12}$$

Let us leave now generalities for a moment and explore possible consequences of this equation for the Z_3 model. In this model there is a nonzero three-point correlator $\langle\epsilon\bar{\sigma}\sigma\rangle$. The general form of this correlator is fixed by the "small" conformal group[5]

$$\langle\epsilon(z_1)\bar{\sigma}(z_2)\sigma(z_3)\rangle \sim \frac{1}{(z_1-z_2)^{\Delta_\epsilon}(z_1-z_3)^{\Delta_\epsilon}(z_2-z_3)^{2\Delta_\sigma-\Delta_\epsilon}} \tag{13}$$

Suppose, on the other hand, that $\epsilon(z)$ is such an operator that $\epsilon^{(-2)}(z)$ in the expansion (10) reduces to $C_2\partial^2\epsilon(z)$, so that the equation (12) is valid. To check this conjecture, put (13) into (12). We obtain then an equation for the critical dimensions:

$$\Delta_\sigma = \frac{\Delta_\epsilon - \Delta_\epsilon^2}{2(2\Delta_\epsilon + 1)} \tag{14}$$

Take for Δ_ϵ the numerical value obtained from the critical exponents $\alpha = 1/3$, $\beta = 1/9$ of the HHM.[4] Standard scaling relations give $(\Delta_\epsilon)_{ph} = 4/5$, $\Delta_\epsilon = 2/5$. Then from (14) we find $\Delta_\sigma = 1/15$, which is in agreement with $(\Delta_\sigma)_{ph} = 2/15$ for HHM.

Note that for IM values of $\Delta_\epsilon = 1/2$, $\Delta_\sigma = 1/16$ equation (12) will also be satisfied by the correlator (13) if we take σ as the operator ψ_1 in (12) instead of ϵ. For the Z_3 model this is not true, which implies that in the expansion (10) for $T\sigma$ the operator $\sigma^{(-2)} \neq C_2\partial^2\sigma$. This only means that the conformal representation of the Z_3 model will be less trivial than that of IM.

This representation will shortly be given, but first let us remark on one intuitively appealing way to derive all the indices of the Z_3 model, not just a relationship among them. The Z_3 model is self-dual, and has a dual (disorder) operator μ. The operator ψ, which results from the product $\sigma_\mu \sim \psi$, has properties of a parafermion with spin $1/3$.[10] In particular it is easily checked in the lattice theory that the correlator $\langle \psi(z)\bar{\mu}(z_1) \ldots \rangle$ is threefold defined—it acquires a factor $\exp(i2\pi/3)$ as z goes round z_1. For $\psi\bar{\mu}$ we have an operator algebra relation:

$$\psi(z)\,\bar{\mu}(z_1) \sim \frac{1}{(z-z_1)^{\Delta_\psi}} \cdot \sigma(z_1) + \cdots \tag{15}$$

Here we use $\Delta_\mu = \Delta_\sigma$. Using this relation for the correlator $\langle \psi\bar{\mu} \ldots \rangle$ we obtain

$$\langle \psi(z)\,\bar{\mu}(z_1) \ldots \rangle_{z \to z_1} \sim \frac{1}{(z-z_1)^{\Delta_\psi}} \langle \sigma(z_1) \ldots \rangle \tag{16}$$

We must have a factor $\exp(i2\pi/3)$ as z goes round z_1. It fixes $\Delta_\psi = 1/3$, at least as a very probable value. Note that we essentially used here that the spin-$1/3$ operator ψ should not depend on $\bar{z}$ (similar to IM fermion[9]) so that $\Delta_\psi = (\Delta_\psi)_{\text{ph}}$, ($\bar{\Delta}_\psi = 0$), and so there is no additional factor $1/(\bar{z} - \bar{z}_1)^{\bar{\Delta}_\psi}$ in (16) for the physical correlator that we have on the lattice.

Operator ψ is created in the product $\sigma\mu$, similar, in a sense, to operator $\epsilon : \sigma\bar{\sigma} \to \epsilon$; see (2′). Suppose that ψ, the same as ϵ, satisfies Eq. (12). Applying it to correlator $\langle \psi\bar{\mu}\bar{\sigma} \rangle$ we derive an equation:

$$\Delta_\sigma = \frac{\Delta_\psi - \Delta_\psi^2}{2(2\Delta_\psi + 1)} \tag{14′}$$

For $\Delta_\psi = 1/3$ we find $\Delta_\sigma = 1/15$, which is the correct value. For fixed $\Delta_\sigma = 1/15$ the equation (14′) has a second solution for Δ_ψ, which is of course $\Delta_\epsilon = 2/5$, corresponding to correlator $\langle \epsilon\bar{\sigma}\sigma \rangle$ considered above.

This way of building a conformal algebra for the Z_3 model looks most natural. Yet certain difficulties arise (which we have not overcome yet) in building the full representation of the algebra that would include $\sigma, \bar{\sigma}$, μ, $\bar{\mu}$, ϵ, $u \sim \sigma\bar{\sigma} - \bar{\sigma}\sigma$, $\psi \sim \sigma\mu$, $\bar{\psi} \sim \bar{\sigma}\bar{\mu}$, $\tilde{\psi} \sim \sigma\bar{\mu}$, $\bar{\tilde{\psi}} \sim \bar{\sigma}\mu$. Perhaps these difficulties are related to the degeneracy among the basic operators ($\sigma\bar{\sigma}\mu\bar{\mu}$ have the same dimension Δ_σ). So in this paper we shall stick to the simplest nondegenerate subalgebra of the Z_3 model—that made by ϵ and $S = \sigma + \bar{\sigma}$. For these operators we should expect the following relations [cf. (2′)]:

$$SS \sim I + S + \epsilon + \cdots, \qquad \epsilon S \sim S + \cdots, \qquad \epsilon\epsilon \sim I + \cdots \tag{2″}$$

Let us return now to the expansion (10). As remarked above in the Z_3 model $S^{(-2)} \neq C_2 \partial^2 S$. We have to look at higher-level operators [operators standing at higher powers of $(z - z_1)$ in expansion (10)] in the search for similar relations on higher levels which would make it possible to derive a closed equation on correlators with S. There is a formal technique for doing this.[6] Let us expand $T(z)$ in powers of $(z - z_1)$

$$T(z) = \sum \frac{L_n(z_1)}{(z - z_1)^{n+2}} \tag{17}$$

and put it into (10). We obtain

$$T(z)\psi(z_1) = \sum_n \frac{1}{(z - z_1)^{n+2}} \psi^{(n)}_{(z_1)} \tag{18}$$

Here by definition $\psi^{(n)} = L_n \psi$. From the WI (12) it follows that

$$L_n \psi = 0, \qquad n > 0, \qquad L_0 \psi = \Delta \cdot \psi, \qquad L_{-1}\psi = \partial \psi \tag{19}$$

Operators $\psi^{(-n)} = L_{-n}\psi$, $n \geqslant 2$ are, in general, new ones, created from the basic operator ψ by conformal transformations. The Nth level linear space of such "conformal followers" of ψ contains operators

$$\begin{gathered} \psi^{(-n_1, -n_2 \ldots -n_k)} = L_{-n_1} L_{-n_2} \cdots L_{-n_k} \psi \\ n_1 < n_2 < \cdots < n_k, \qquad \sum_{i=1}^{k} n_i = N \end{gathered} \tag{20}$$

Operators $\{L_n\}$ do not commute. They form the Virasoro algebra[11]

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{C}{12}\delta_{n,-m} \cdot n(n^2 - 1) \tag{21}$$

The number C here is a coefficient in $\langle T(z)T(z')\rangle = (C/2)/(z - z')^4$ and it is a fundamental parameter of a particular conformal algebra, togther with the dimensions $\{\Delta_i\}$ of its basic operators.

In general the operators (20) are linearly independent. But there are special basic operators ψ, whose dimension Δ satisfies the equation:[12]

$$\left[\Delta + \frac{(n^2 - 1)(C - 13)}{24} + \frac{nm - 1}{2}\right] \cdot \left[\Delta + \frac{(m^2 - 1)(C - 13)}{24} + \frac{nm - 1}{2}\right] + \frac{(n^2 - m^2)^2}{16} = 0 \tag{22}$$

for which not all of the operators (20) are independent. In (22) n, m are any integer numbers such that $n \cdot m = N$ (on higher levels there can be several degenerate linear combinations). It can be shown, in particular, that correlators with such special operators satisfy the Nth order linear differential equation.[6]

Let us solve (22) for Δ. We find

$$\Delta_{n,m} = \frac{(\gamma n - \gamma^{-1} m)^2 - (\gamma - \gamma^{-1})^2}{4}$$
$$\gamma = \left\{2\left[\xi - (\xi^2 - 1/4)^{1/2}\right]\right\}^{1/2}, \qquad \xi = (13 - C)/24 \tag{23}$$

Now we look at the Z_3 algebra. We already know that a "conformal tail" [states (20)] of the operator ϵ is degenerate on the second level. In fact, $\epsilon^{(-2)} = C_2 \partial^2 \epsilon$, which means that $L_{-2}\epsilon - C_2 L_{-1}^2 \epsilon = 0$ [see (19)]. On the second level Eq. (22) becomes

$$C = \frac{2\Delta(5 - 8\Delta)}{2\Delta + 1} \tag{24}$$

Putting $\Delta_\epsilon = 2/5$, we find for the Z_3 conformal algebra $C = 4/5$. Then (23) becomes

$$\gamma = (5/6)^{1/2}, \qquad \Delta_{n,m} = \frac{(5n - 6m)^2 - 1}{120} \tag{25}$$

This set of dimensions is given in Table I. Notice that all the operators are doubly represented in Table I, so that we have in fact 10 basic conformal operators.

Let us look at their algebraic relations. Consider a correlator $\langle \psi_{1,2}(z) \psi_{n,m}(z') \ldots \rangle$. This correlator satisfies Eq. (12) for z. Take $z \to z_1$

$$\langle \psi_{1,2}(z) \psi_{n,m}(z') \ldots \rangle \sim \frac{1}{(z - z')^{\Delta_{1,2} + \Delta_{n,m} - \Delta_A}} \langle A(z') \ldots \rangle$$

Table I.

5	3	$\frac{7}{5}$	$\frac{2}{5}$	0
4	$\frac{13}{18}$	$\frac{63}{120}$	$\frac{1}{40}$	$\frac{1}{8}$
3	$\frac{2}{3}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{3}$
2	$\frac{1}{8}$	$\frac{1}{40}$	$\frac{63}{120}$	$\frac{13}{18}$
1	0	$\frac{2}{5}$	$\frac{7}{5}$	3
	1	2	3	4

and put it into Eq. (12). Then most singular terms give the characteristic equation for Δ_A. Solving it we find $\Delta_A = \Delta_{n,m\pm1}$. Thus we have

$$\psi_{1,2}\psi_{n,m} \sim \psi_{n,m+1} + \psi_{n,m-1} \tag{26}$$

In a similar way, the following relations can be derived:

$$\begin{aligned}
&\psi_{1,3}\psi_{n,m} \sim \psi_{n,m} + \psi_{n,m+2} + \psi_{n,m-2} \\
&\psi_{3,2}\psi_{n,m} \sim \psi_{n,m+1} + \psi_{n,m-1} + \psi_{n+2,m+1} + \psi_{n+2,m-1} \\
&\qquad + \psi_{n-2,m+1} + \psi_{n-2,m-1} \\
&\psi_{3,3}\psi_{n,m} \sim \psi_{n,m} + \psi_{n,m+2} + \psi_{n,m-2} + \psi_{n+2,m} + \psi_{n+2,m+2} + \psi_{n+2,m-2} + \cdots
\end{aligned} \tag{27}$$

and so on. Looking now at Table I we find that the physical relations (2″) for operators (ϵ, S) are fulfilled. We also find that operators do not couple to four operators in the second row, so that we are left with six basic operators, those in Table II.

Now we just list some results for the simplest multipoint correlation functions of the Z_3 model:

$$\langle \psi_{n,m}(z_1)\epsilon(z_2)\psi_{n,m}(z_3)\epsilon(z_4)\rangle \sim a_1 w_1(\alpha, \beta, \gamma; \eta) + a_2 w_2(\alpha, \beta, \gamma; \eta) \tag{28}$$

$$\langle \epsilon_1\epsilon_2\epsilon_3\epsilon_4\rangle = \left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}}\right)^{4/5} \cdot \left[a_i w_1\left(-\frac{8}{5}, -\frac{1}{5}, -\frac{2}{5}; \eta\right) + a_2 w_2(\ldots, \eta)\right] \tag{29}$$

$$\langle s_1\epsilon_2 s_3\epsilon_4\rangle = \frac{(z_{13})^{2/3}}{(z_{12}z_{23}z_{34}z_{14})^{2/5}} \cdot \left[a_1 w_1\left(-\frac{4}{5}, \frac{3}{5}, \frac{2}{5}; \eta\right) + a_2 w_2(\ldots; \eta)\right] \tag{30}$$

Here $\eta = (z_{12}z_{34})/(z_{13}z_{24})$; $\psi_{n,m} = X, Y, Z$; the functions $w_1(\ldots; \eta)$,

Table II.

5	Y	X	ϵ	I
4				
3	Z	S	S	Z
2				
1	I	ϵ	X	Y
	1	2	3	4

$w_2(\ldots;\eta)$ are two solutions of the hypergeometric equation with parameters

$$\alpha_{n,m}=\frac{5(n+1)-6(m+1)}{5}, \qquad \beta_{n,m}=\frac{5n-6(m-1)}{5}$$

$$\gamma_{n,m}=\frac{5(n+1)-6m}{5}$$

To obtain the physical correlators, depending on $\{z_i,\bar{z}_i\}$, we just choose any conformal solution $w(\eta)$ in (28), multiply it by its complex conjugate $\bar{w}(\eta)$ and sum the resulting product over a monodromy group[13] of $w(\eta)$:

$$\langle\{z_i\}\{\bar{z}_i\}\rangle=\frac{1}{[z_i]\cdot[\bar{z}_i]}\cdot\sum_m w_{(gm)}(\eta)\cdot\overline{w_{(gm)}}(\eta) \tag{31}$$

A simple example of this technique gives the correlator $\langle\sigma\sigma\sigma\sigma\rangle$ of the IM. As found in Ref. [6] the conformal solution for this correlator is given by

$$\langle\sigma\sigma\sigma\sigma\rangle=\left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}}\right)^{1/8}\cdot w(\eta)$$

Here $w(\eta)=[1+(1-\eta)^{1/2}]^{1/2}$, and it has a monodromy around $\eta=1$; $g_1:[1+(1-\eta)^{1/2}]^{1/2}\to[1-(1-\eta)^{1/2}]^{1/2}$, $(g_1)^2=1$. In this case (31) becomes

$$\begin{aligned}\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle &= \left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{15}}\right|^{1/4} \\ &\quad\cdot\left\{\left[1+(1-\eta)^{1/2}\right]^{1/2}\left[1+(1-\bar{\eta})^{1/2}\right]^{1/2}\right. \\ &\quad\left.+\left[1-(1-\eta)^{1/2}\right]^{1/2}\left[1-(1-\bar{\eta})^{1/2}\right]^{1/2}\right\} \\ &\sim\left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}}\right|^{1/4}\cdot\left\{1+(\eta\bar{\eta})^{1/2}\left[(1-\eta)(1-\bar{\eta})\right]^{1/2}\right\}^{1/2} \\ &=\left[\left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}}\right|^{1/2}+(3\leftrightarrow 4)+(2\leftrightarrow 3)\right]^{1/2}\end{aligned} \tag{32}$$

which coincides with the known result.[14] Application of this technique to the Z_3 correlators is straightforward. This work is under way now.

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